12038. Proposed by George Apostolopoulos, Messolonghi, Greece.

Let ABC be an acute triangle with sides of length $a, b,$ and c opposite angles $A, B,$ and $C,$ respectively, and with medians of length $m_a, m_b,$ and m_c emanating from $A, B,$ and $C,$ respectively. Prove

$$\frac{m_a^2}{b^2+c^2}+\frac{m_b^2}{c^2+a^2}+\frac{m_c^2}{a^2+b^2} \geq 9\cos A\cos B\cos C.$$

Solution by Arkady Alt, San Jose, California, USA.

Since* $m_a \geq \dfrac{b^2+c^2}{4R}$ then $\sum_{cyc}\dfrac{m_a^2}{b^2+c^2} \geq \sum_{cyc}\dfrac{b^2+c^2}{16R^2} = \dfrac{a^2+b^2+c^2}{8R^2} = \dfrac{s^2-4Rr-r^2}{4R^2}$

Also note that $\cos A\cos B\cos C = \dfrac{s^2-(2R+r)^2}{4R^2}$. Thus, suffices to check inequality $s^2-4Rr-r^2 \geq 9\left(s^2-(2R+r)^2\right)$.

Since $R \geq 2r$ (Euler′s Inequality) and $s^2 \leq 4R^2+4Rr+3r^2$ (Gerresen′s Inequality) we obtain

$s^2-4Rr-r^2-9\left(s^2-(2R+r)^2\right) = 4(9R^2+8Rr+2r^2-2s^2) =$

$4(R-2r)(R+2r)+8(4R^2+4Rr+3r^2-s^2) \geq 0.$

*Let R and d_a be, respectively, circumradius and distance from the circumcenter to side a. Then by triangle inequality $|m_a-R| \leq d_a$ and, since $d_a = \sqrt{R^2-\dfrac{a^2}{4}}$ then we obtain $|m_a-R| \leq \sqrt{R^2-\dfrac{a^2}{4}} \Leftrightarrow m_a^2-2m_aR+R^2 \leq R^2-\dfrac{a^2}{4} \Leftrightarrow$

$4m_a^2-8m_aR+a^2 \leq 0 \Leftrightarrow 2(b^2+c^2)-a^2-8m_aR+a^2 \leq 0 \Leftrightarrow b^2+c^2 \leq 4m_aR \Leftrightarrow$

$m_a \geq \dfrac{b^2+c^2}{4R}.$